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Alexis Blasselle
Tek Diving

Michaël Theron
Université de Bretagne Occidentale

Emmanuel Dugrenot
DAN - Divers Alert Network

A new form of admissible pressure for Haldanian decompression models

Alexis Blasselle[a,*], Michael Theron[b], Bernard Gardette[a], Emmanuel Dugrenot[a,b]

[a] *Tek Diving SAS, Brest, France*
[b] *ORPHY laboratory, University of Western Brittany, IBSAM, Brest, France*

Abstract

In this article, we propose and study a new form of admissible pressure in the Haldanian framework. We then use it to study the surjectivity of the Gradient Factors on the space of the reachable decompression profiles, and investigate a particular case. This case leads to the proposition of a decompression strategy, whose crucial parameter is the ascent rate. An appropriate ascent rate is suggested as recommended by COMEX, through a physiologically relevant method. This new strategy enables the unification of the COMEX approach (not based on a tissue saturation theory) with the Haldanian method.

Keywords: Scuba diving, Decompression sickness, Haldanian models, Admissible pressure, Personalized decompression

1. Introduction

Scuba diving is a recreational activity practiced by approximately 7 million regular divers. In addition to those who practice recreationally, there are approximately 10,000 diving professionals, including military and civilian underwater workers. At the end of an immersion, divers have to manage their return to the surface using protocols that can include empirical tables (ultimately based on mathematical models) or models implemented in dive computers.

*Corresponding author
Email address: `alexis@tek-diving.fr` (Alexis Blasselle)

The so-called Bühlmann models are among the most frequently used models in dive computers. They were derived from the work of Dr. Albert A. Bühlmann (physician at the University of Zurich), and published for the first time in 1984 [1]. Although initally criticized by technical divers (interested in helium diving), the Bühlmann *model* and its derivatives are now widely used, due to the Gradient Factors introduced by Erik C. Baker in 1998 [2], which enriched the initial model and improved its flexibility.
Although satisfactory, Bühlmann models do not take into account the individual characteristics of their users and epidemiological and physiological approaches provide clear evidences of inter and intra-individual variations in decompression sickness (DCS) susceptibility. For instance it is now clear that rats have eritable determinants of decompression sickness ([3]) and some divers produce large amounts of bubbles during decompression while others do not ([4], [5], [6]). Adapting the settings of the decompression algorithms to the divers' characteristics with regard to known risk factors is relevant.

To introduce more flexibility in the decompression calibration, given the limitations of mathematical models alone, we propose a dual approach based on:

- A calibration of the Bühlmann model using empirical protocols such as MN90, COMEX, or MT92 tables in a specific application framework.
- A new form of conservatism that is combined or added to the Gradient Factors.

To achieve this double approach, we developed a new form of admissible pressure and investigated the relevance of a new strategy related to the speed of ascent. We then suggested choosing it according to COMEX protocols from which the French MT92 dive tables are derived. This article is organized as follows : in the first part, we briefly recap the principle of the Haldanian framework and Bühlmann's model, along with the Gradient Factors (GF). In the second part, we introduce the new form of admissible pressure that allows us to study the question raised about the Gradient Factors in the third section. The fourth section is dedicated to the new ascent strategy. In the fifth and final sections, we discuss of the importance of the ascent speed.

2. Notations and framework

We present in this section the general notations used in the article and briefly remind Haldane's assumption and Bühlmann's model.

2.1. Notations

Please note that some specific notations may be used in some paragraphs.

- D is the depth
- P is the ambient pressure
- Q is the ambient pressure admissible by the diver
- A is the inert gas pressure in one general compartment
- N is the number of compartments
- A_i is the inert gas pressure in the the compartment $i = 1...N$
- Q_i is the ambient pressure admissible by the compartment i
- R is the inert gas ratio
- P_S is the surface pressure
- T is a general fixed time
- t_f is the final time of a given dive
- CH is the sum of the partial pressures in all of the compartments at the end of the dive
- K is a compact of $\mathbb{R}_+$ of the form $[0, T]$

- t is the current time
- v is the diver's vertical speed (rate of ascent or descent)
- l and h are the low and high Gradient Factors, respectively
- τ and τ_i are the characteristic time of one general compartment and the characteristic time of compartment i, respectively.

We make the following regularity hypothesis of the diving profiles:

Assumption 1. *All diving profiles P are at least twice differentiable with respect to time, and their second-order derivatives are continuous. They have a finite compact support K contained in $\mathbb{R}_+$. Because they are continuous over a compact, they are bounded. Finally, they are always positive.*

We use the term "rectangular profile" for a dive constituted by a straight descent followed by a phase spent at a fixed depth. To simplify the framework, we also make the following assumption:

Assumption 2. *We study open circuit dives, with one inert gas ratio R. All of the results are applicable to*

- *the semi-closed and closed circuit cases, by simply using the oxygen partial pressure in the equations*
- *the multiple inert gas cases, by simply using the partial ratios in the equations.*

We use Haldane's and Bühlmann's framework and a parallel compartment model. As this is the most common model used in the majority of diving computers, we propose slight and incremental modifications, while using its flexibility to approximate empirical protocols.

2.2. The existing framework

Haldane assumed that the body is divided into N compartments. We consider 16 for the numerical applications in this article, but nothing changes if we increase or decrease this number. If A is the inert gas pressure in any compartment, Haldane formulated that this pressure is linked to the external pressure by the equation:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = k\,(RP - A), \tag{1}$$

where k is a real constant, and the initial condition is $A(0) = A_0$. Note that k satisfies:

$$k = \frac{\ln 2}{\tau}$$

We solve equation (1) to obtain:

$$A(t) = A_0 e^{-kt} + kR \int_0^t P(\nu)\exp(-k(t-\nu))\,d\nu \tag{2}$$

See [7] for more details. When the pressure is constant over the dive, that is, when $P(s) = \hat{P}$:

$$A(t) = A_0 + (R\hat{P} - A_0)(1 - 2^{-t/\tau}) \tag{3}$$

Bühlmann's work proposed a form for the minimum admissible pressure Q_i of each compartment i: $\forall\, t \in K$:

$$Q_i(t) = b_i\,(A_i(t) - a_i) \tag{4}$$

For the global minimum admissible pressure

$$Q(t) = \max_{i=1..N} Q_i(t) \tag{5}$$

The Gradient Factors were defined by Eric C. Baker in [2]. They are widely used, especially by technical divers. This study was initiated by a question that will be addressed in a later section. The two Gradient Factors l and h define a slope Δ:

$$\Delta = \frac{l - h}{D_M - D_1},$$

where D_M is usually the maximum depth reached, and D_1 is the last stop, generally 0. The global Gradient Factor G is:

$$G(t) = h + \Delta\, D(t) \qquad \forall\, t \in K$$

and the new admissible pressure is expressed as follows for a general compartment:

$$Q(t) = \frac{A(t) - G(t)\, a}{\dfrac{G(t)}{b} - G(t) + 1} \qquad \forall\, t \in K \tag{6}$$

Remark 1. *As D_M is the maximum depth reached during a dive, for a rectangular profile, throughout the descent and planar phase, $G = l$.*

2.3. Preliminary property

A simple but useful property is now described. Starting from the equation with Q_i as a function of A_i

$$Q_i = \frac{A_i - a_i(h + \alpha(l-h)D)}{1 + \left(\frac{1}{b_i} - 1\right)(h + \alpha(l-h)D)} \qquad \text{with} \qquad \alpha = \frac{1}{D_M}$$

and because

$$a_i(h + \alpha(l-h)D) \,>\, 0 \qquad \text{and}$$
$$0 \,\leq\, h + \alpha(l-h)D \,\leq\, 1$$

then

$$A_i - a_i(h + \alpha(l-h)D) \,<\, A_i \qquad \text{and}$$
$$1 + \left(\frac{1}{b_i} - 1\right)(h + \alpha(l-h)D) \,\geq\, 1$$

resulting in

$$\forall\, i = 1..N \qquad \forall\, t \,\geq\, 0 \qquad Q_i(t) \,<\, A_i(t) \tag{7}$$

3. New form of admissible pressure

3.1. Justification and definition

In Bühlmann model, as expressed by equation (5), the minimum admissible pressure is the maximum value calculated across the admissible pressure over all of the compartments. Regarding this choice:

- We consider only the compartment that matters the most at a given time but do not directly account for the computations of the other compartments, even if their admissible pressure is as close as possible to the one reaching the maximum,
- As those compartments are not real and physiological compartments in the body, it seems rigid to use such an abrupt assumption and not directly consider in the computation of the minimum admissible pressure the compartments whose admissible pressure is not the greatest.

This last point is of paramount importance: we work with theoretical models, and fixing in them strict mathematical boundaries may be somewhat harsh. Indeed, by considering only the leading compartment, we may not optimize the decompression with respect to some of, or all the other ones. Therefore, in a different way we consider all of the compartments in the computation of the minimum admissible pressure.

The maximum in equation (5) is an N_∞ norm that is the limit of the N_p norm when p tends towards $+\infty$, it appears natural to suggest the new kind of admissible pressure defined herein.

Proposal 1. *We propose a new form of minimum admissible pressure, depending on the admissible pressures in each compartment:* $\forall\, t \in K$

$$\boxed{Q(t) = \left(\sum_{i=1}^{N} Q_i(t)^p \right)^{\frac{1}{p}}} \tag{8}$$

Due to assumption 1 and equation (2)*, we know that this quantity is defined, and has at least the same regularity properties as* P*.*

3.2. Some numerical results

This section presents some results to show the form of the new proposed admissible pressure. We consider a random test dive, approximately 55m deep, with GF 0.3/0.45, which was the GF selected by the diver after a DCS that occurred during the dive. The following four values are used for parameter p:

- $p = 80$ as close as possible to Bühlmann's original algorithm
- $p = 16$ not conservative
- $p = 12$ mildly conservative
- $p = 7$ very conservative

We use $p = 80$ only for validation purposes for the first GF to show that we can get as close as possible to Bühlmann original algorithm, and will not use it afterwards. Figure 1 is a simulation of a Bühlmann classical decompression ceiling for GF 0.3/0.45 in a thin green line, and $p = 80$ for the same GF in thick blue line. The 2 curves are perfectly identical. BU represents Bühlmann and TD denotes the new proposed form of minimum admissible pressure.

Appendix B shows simulations of various Gradient Factors.

In the graphs that follow, the ascent phase is highlighted as we are interested in the decompression. This new form of minimum admissible pressure, for a couple of higher Gradient Factors, crosses the Bühlmann nominal admissible pressure for the hard GF couple of 0.3/0.45, as shown in this example. Hence, this new way of computing the admissible pressure is another method of making the decompression profile more conservative, but different from the technique that the Gradient Factors enable.

In the following tables, the new minimum admissible pressure is tested on rectangular profiles, along with the pure Bühlmann algorithm, with and without Gradient Factors. For a given p leading to the duration of a dive, we found a set of Gradient Factors leading to the same duration. We also studied the total sum of the compartments' partial pressures at the end of the dive, namely

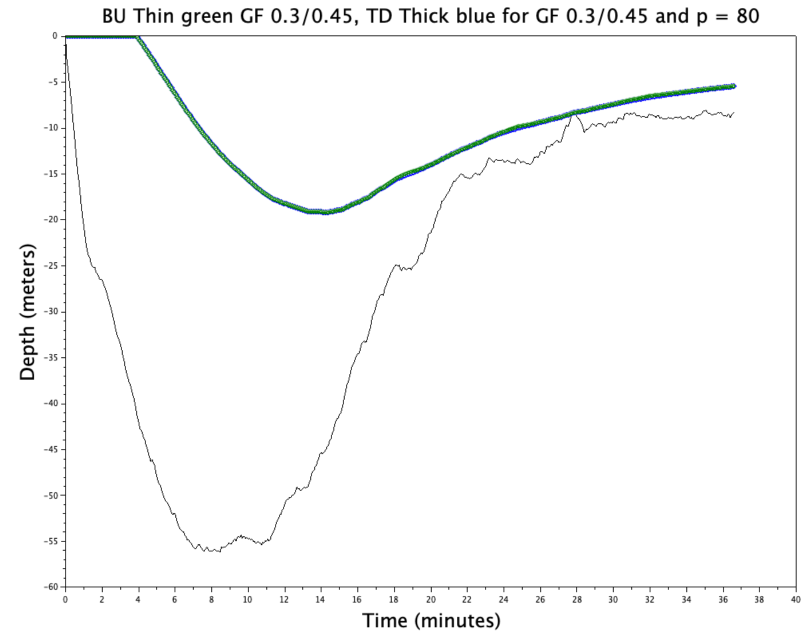

Figure 1: Diving profile in a thin black line and pure Bühlmann decompression ceiling in a thin green line, indiscernible from our proposed decompression ceiling of $p = 80$ in a thick blue line, both for GF 0.3 / 0.45

$$CH = \sum_{i=1}^{N} A_i(t_f)$$

Of note:

- For a given dive duration t_f, the proposed new form tends to add time to decompression stops that are closer to the surface compared to those added by some Gradient Factors
- for a given dive duration t_f, the proposed new form lowers the sum CH of the partial pressures in the compartments at the end of the dive.

Hence, our new minimum admissible pressure does not replace the Gradient Factors, but can be used with them, as it differently modulates the decompression. The new proposed form of admissible pressure is smoother and can be

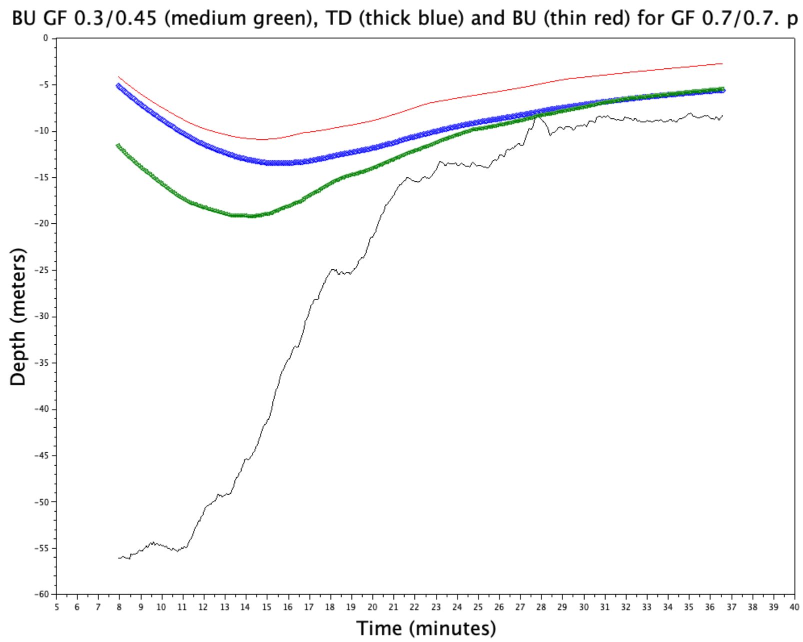

Figure 2: Thin red line: Bühlmann GF 0.3/0.45, Thick blue line: Bühlmann GF 0.7/0.7, Medium green line: new pressure $p = 7$ GF 0.7/0.7

implemented more easily than the pure Bühlmann's algorithm, which requires elucidating the maximum at each time step.

3.3. Condition over p

This section considers any dive of maximum pressure P_M and duration T. The maximum value that the inert gas pressure of any compartment can reach is RP_M. This can easily be verified using equation (3). A boundary over $Q(T)$ must be found. Starting from equation (8) and using equation (7) and the fact

D = 35 meters for T = 40 minutes								
P	GF or value of p	CH	Duration	15	12	9	6	3
Bü	1.0 / 1.0	20.253	73.1			1	9	22
TD	p = 80	20.150	74.1			1	9	23
TD	p = 50	20.078	75.1			1	10	23
TD	p = 30	19.934	77.1			2	10	24
Bü	0.96 / 0.96	19.934	77.1			2	10	24
TD	p = 20	19.634	81.1			3	10	27
Bü	0.92 / 0.92	19.634	81.1			3	10	27
TD	p = 16	19.450	84.1			3	12	28
Bü	0.87 / 0.9	19.467	84.1			4	11	28
TD	p = 12	18.964	93.1			5	13	34
Bü	0.81 / 0.81	19.017	93.1			6	14	32
TD	p = 11	18.769	97.1			5	14	37
Bü	0.77 / 0.78	18.850	97.1			7	15	34
TD	p = 10	18.489	104.1			6	15	42
Bü	0.72 / 0.73	18.573	104.3		1	8	17	38
TD	p = 9	18.163	114.1			7	17	49
Bü	0.68 / 0.68	18.274	114.3		2	9	19	44
TD	p = 7	17.157	168.3		2	10	24	92
Bü	0.35 / 0.51	17.450	168.5	3	8	14	30	74

Figure 3: Comparison of the new admissible pressure (TD) and classical Bühlmann (BU) with Gradient Factors for a rectangular profile D=35m for T=40 minutes

that $\forall\, i = 1..N,\ A_i(T) \le RP_M$:

$$Q(t)^p = \sum_{i=1}^{N} Q_i(t)^p$$

$$Q(t)^p < \sum_{i=1}^{N} A_i(t)^p$$

$$Q(T)^p < N R^p P_M^p$$

leading to

$$Q(T) < N^{\frac{1}{p}} R P_M$$

D = 50 meters for T = 25 minutes									
P	GF or value of p	CH	Duration	18	15	12	9	6	3
Bü	1.0 / 1.0	20.268	61.8			1	5	8	21
TD	p = 80	20.161	62.8			1	5	8	22
TD	p = 50	20.086	63.8			1	5	9	22
TD	p = 20	19.709	68.8			2	5	10	25
Bü	0.93 / 0.93	19.709	68.8			2	5	10	25
TD	p = 16	19.453	72.8			2	6	11	27
Bü	0.87 / 0.9	19.468	72.8			3	5	11	27
TD	p = 12	19.008	80.8			3	6	13	32
Bü	0.81 / 0.81	19.092	80.8			4	7	13	30
TD	p = 11	18.792	85.8			4	6	14	35
Bü	0.788 / 0.788	18.880	85.0		1	4	7	14	33
Bü	0.78 / 0.78	18.845	86.0		1	4	7	15	33
TD	p = 10	18.546	91.8			4	7	15	39
Bü	0.75 / 0.75	18.684	91.0		2	4	7	17	35
Bü	0.74 / 0.74	18.637	92.0		2	4	7	17	36
TD	p = 9	18.203	101.0		1	4	7	16	47
Bü	0.695 / 0.695	18.365	101.0		3	4	8	19	41
TD	p = 7	17.188	154.0		2	5	10	23	88
Bü	0.48 / 0.51	17.512	154.2	3	5	8	13	29	71

Figure 4: Comparison of the new admissible pressure (TD) and classical Bühlmann (BU) with Gradient Factors for a rectangular profile D=50m for T=25 minutes

Hence, choosing p such that $N^{\frac{1}{p}}R < 1$ obtains $Q(T) < P_M$.

Property 1. *For any dive of maximum pressure P_M and duration T, choosing p such that*

$$p > -\frac{\ln N}{\ln R}$$

ensures that

$$Q(T) < P_M$$

A numerical application for $N = 16$ and $R = 0.79$ obtains $p \geq 12$.

3.4. Further steps to introduce variability

Using the exact same algorithm for everyone is not a realistic assumption, as people differ and are not equal with respect to decompression sickness. Hence, introducing variability leverages in the models is necessary to personalize the protocols. These leverages are not manipulated by the divers themselves, as they are complex to elucidate and program in a diving computer, but in the future, a physiological individualized approach will enable engineers and physiologists to personalize a diver's computer with these variabilities.

Multivariate p. Starting from the new proposed admissible pressure defined in equation 8, we propose the following form: $\forall\, t \in K$

$$Q(t) = \left(\sum_{i=1}^{N} Q_i(t)^{p_i} \right)^{\frac{1}{p}} \qquad \text{with} \qquad p = \sum_{i=1}^{N} p_i$$

This enables variable modulation over each compartment, depending on their relative importance.

Higher degree polynomials. Equation (4) expresses a linear dependency between the pressure admissible by a compartment and its inert gas pressure. This dependency is a first degree polynomial, but a higher degree dependency could be tried, with appropriate coefficients. This is expressed as

$$Q_i(t) = \sum_{k=0}^{s} c_{i,k} A_i^k(t) \tag{9}$$

Again, this enables more possibilities for introducing variability, for example an empirical protocol with an Haldanian approach.

Remark 2. *In this article, we do not suggest changing the coefficients a_i, b_i and τ_i proposed by Bühlmann, because, for the time being, we want to be able to return to his original protocol, as it is the most widely implemented method in diving computers. Parameter p could be a simple enough variable for divers to set and adjust directly, such as the Gradient Factors, which many divers* *set themselves on their own computers. Nevertheless, coefficients a_i, b_i and τ_i should be considered as well as optimization and personalization leverages.*

The underlying concept for all the new leverages is to be able to set them for each diver, using his or her physiological characteristics. No method is provided in this article to directly find the best values for p or the other proposed coefficients, but we are working on such methods, and already designed one taking into account physiological parameters. This will be the subject of a future article.

4. Surjectivity of the Gradient Factors

This section answers a question asked by a scuba diver: could we generate any decompression profile using the two previously presented Gradient Factors?

4.1. Framework

$\mathcal{M}(l, h)$ is the function that associates, to a given depth dive profile $D : t \mapsto D(t)$, the corresponding admissible pressure profile $\hat{Q} : t \mapsto \hat{Q}(t)$ for the Gradient Factors $(l,\, h)$

$$\mathcal{M}(l,\, h) :\, D \mapsto \hat{Q}$$

Whether or not we can generate any possible decompression profile using the two Gradient Factors l and h can be reformulated as follows: for every dive profile D and every admissible pressure $\hat{Q}$, is there a couple $(\hat{l},\, \hat{h})$ verifying

$$\mathcal{M}(\hat{l},\, \hat{h})(D) = \hat{Q}$$

4.2. Surjectivity

For compartment i, equation (6) is

$$Q_i(t) = \frac{A_i(t) - a_i G(t)}{1 + \left(\frac{1}{b_i} - 1\right) G(t)}$$

If we momentarily drop the time dependency to simplify the notations, this can be rewritten

$$Q_i = \frac{A_i - a_i(h + \alpha(l - h)D)}{1 + \left(\frac{1}{b_i} - 1\right)(h + \alpha(l - h)D)} \qquad \text{with} \qquad \alpha = \frac{1}{D_M}$$

The new admissible pressure defined in equation (8) is now used instead of the maximum defined in equation (5) and used by the pure Bühlmann algorithm. Before explicitly looking for $(\hat{l},\, \hat{h})$, the problem is rewritten with the new pressure and studied. Taking the power p, the new admissible pressure is then

$$\boxed{\hat{Q}(t)^p = \sum_{i=1}^{N} \frac{[A_i(t) - a_i(h + \alpha(l-h)D(t))]^p}{[1 + \left(\frac{1}{b_i} - 1\right)(h + \alpha(l-h)D(t))]^p}} \tag{10}$$

Multiplying this by the products of all the denominators obtains:

$$\hat{Q}(t)^p \prod_{i=1}^{N} \left[1 + \left(\frac{1}{b_i} - 1\right)(h + \alpha(l-h)D(t))\right]^p$$
$$= \sum_{i=1}^{N} [A_i(t) - a_i(h + \alpha(l-h)D(t))]^p \prod_{j \neq i} \left[1 + \left(\frac{1}{b_j} - 1\right)(h + \alpha(l-h)D(t))\right]^p$$

A_i is defined by equation (2), with the proper coefficients for the compartment i. We then obtain a $2p$ multinominal equation in $(l,\, h)$. Given any D and any admissible $\hat{Q}$can $(\hat{l},\, \hat{h})$ stay on the parametrized surface defined by this multinomial form? This is too constrained to be satisfied, and the best way to demonstrate it is with a counter example, using the rectangular profile defined by

$$P(t) = \begin{cases} \dfrac{P_M}{t_1} t & 0 \le t \le t_1 \\ \\ P_M & t_1 \le t \le t_2 \end{cases}$$

Equation (2) is used to compute A_i from P and easily obtain: $\forall\, t \le t_1$

$$A_i(t) = A_{i,0} \exp(-k_i t) + P_M R \frac{t}{t_1} - \frac{P_M R}{k_i t_1}(1 - \exp(-k_i t)) \qquad t \le t_1 \tag{11}$$

and $\forall\, t \in [t_1; t_2]$

$$A_i(t) = A_{i,0} \exp(-k_i t) + P_M R + \frac{P_M R}{k_i t_1} \exp(-k_i t)(1 - \exp(k_i t_1)) \qquad t_1 \le t \le t_2 \tag{12}$$

From remark 1, $\forall\, t \le t_2$ and $G = l$, and we use only the low gradient factor l.

Hence, rewriting equation (10) for $t \leq t_2$ obtains the simpler form:

$$\hat{Q}(t)^p \prod_{i=1}^{N} \left[1 + \left(\frac{1}{b_i} - 1\right) l\right]^p = \sum_{i=1}^{N} [A_i(t) - a_i l]^p \prod_{j \neq i} \left[1 + \left(\frac{1}{b_j} - 1\right) l\right]^p \quad (13)$$

Equations (11) and (12) clearly demonstrate that the right-hand side of the equation is a linear combination of exponential terms, multiplied by constant or linear terms in t (there is no case in which all of the exponential terms vanish, as the k_i are different). As a counter example, we consider **any** polynomial form for $\hat{Q}$, with no exponential in it. Even $\hat{Q}(t) = Q_0 \frac{t}{T}$ is a valid choice. Thus, equation (13) can not be satisfied, and we obtain the desired counter example.

Property 2. *For a given diving depth profile D, we cannot generate any admissible pressure Q (decompression ceiling) using only the two Gradient Factors l and h.*

The new proposed form of admissible pressure does not permit the generation of any admissible pressure. However, new generalized and more variant forms such as the one suggested in equation (9) might.

5. A new approach, by iteration

5.1. Decompression strategy

Targeting the decompression ceiling. A random dive of duration T, without the ascent phase, is considered. We assume that, even if the profile is not rectangular and the diver has already begun to ascent, he or she has not crossed his or her decompression ceiling $Q(T)$ at the time T.

Assumption 3. *We consider a dive of duration T, noton necessarily rectangular, and we suppose that $Q(T) < P(T)$. The notation $P_1 = Q(T)$ is used in this paragraph.*

At time T, the diver tries to reach $Q(T)$ with an ascent speed v (in bar/min). What happens to Q from this moment? Even if the diver was saturated at the pressure $P(T)$, the non-trivial form of each Q_i

$$Q_i(t) = \frac{A_i(t) - a_i G(t)}{1 + \left(\frac{1}{b_i} - 1\right) G(t)}$$

prevents us from asserting that Q_i will decrease, and that Q will decrease from T. This is normal: if the ascent is slow, some of the compartments may still load in inert gas during it. We do not know if Q will increase or decrease, and we do not even know if the diver will not **cross his or her decompression ceiling** Q while ascending.

Waiting for Q to decrease. Hence, we propose that the diver reaches P_1 and then, if necessary, waits for the decompression ceiling to decrease *enough* (in a sense defined in what follows) at pressure P_1. As demonstrated by property 5 given in Appendix A, all of the inert gas compartments' pressures A_i have the same limit $RP_1 < P_1$, each reaching it by lower or upper values. Equation (7) demonstrates that

- if A_i reaches RP_1 by increasing, then $\forall\, t \geq T$, $Q_i(t) < RP_1$
- if A_i reaches RP_1 by decreasing, then $\forall\, \varepsilon > 0$, $\exists\, t_{i,\varepsilon}$ such that $\forall\, t > t_{i,\varepsilon}$, we have $A_i(t) - RP_1 < \varepsilon$ and

$$Q_i(t) - RP_1 < \varepsilon$$

From this and the definition of Q, $\forall\, t > \max_i t_{i,\varepsilon}$

$$Q^p(t) = \sum_{i=1}^{N} Q_i(t)^p < N(RP_1 + \varepsilon)^p$$

leading to

$$Q(t) < N^{\frac{1}{p}} RP_1 + \varepsilon'$$

with ε' defined as a function of ε. Since we chose p such that $N^{\frac{1}{p}} R < 1$, we choose a positive real λ such that

$$N^{\frac{1}{p}} R < \lambda < 1 \tag{14}$$

Here is the important part of the reasoning: we can use the smallest ε that we want, and especially so

$$N^{\frac{1}{p}} R P_1 + \varepsilon' \, < \, \lambda P_1$$

This is summed up in the following property

Property 3. *λ is a fixed positive real number satisfying*

$$N^{\frac{1}{p}} R \, < \, \lambda \, < \, 1$$

and we consider a random dive of duration T without the ascent process, and such that the assumption 3 holds. We suppose that, from the time T, the diver reaches $Q(T)$ with an ascent speed $v \, > \, 0$, and then stays at the pressure $Q(T)$ as long as needed. There is a finite time $\hat{t}$ such that, $\forall \, t \, > T + \hat{t}$

$$Q(t) \, < \, \lambda \, Q(T)$$

5.2. The recurrence defining the strategy

The profile alternates between linear ascent phases and planar phases, to give Q enough time to sufficiently decrease if necessary. Hence, λ is fixed and satisfies equation (14). We define the steps of the sequence from T, for $n \, \in \mathbb{N}$ and while $Q(t_n) \, \geq \, P_S$ as follows:

$$\begin{cases} P(t) = & P(t_n) - v_n(t - t_n) \qquad t_n \, \leq \, t \, \leq \, t_n + \hat{t}_{n,1} \\ P(t) = & Q(t_n) \qquad\qquad t_n + \hat{t}_{n,1} \, \leq \, t \, \leq \, t_n + \hat{t}_{n,1} + \hat{t}_{n,2} \\ t_{n+1} = & t_n + \hat{t}_{n,1} + \hat{t}_{n,2} \\ t_0 = & T \\ \hat{t}_{n,1} = & \dfrac{P(t_n) - Q(t_n)}{v_n} \\ \hat{t}_{n,2} = & \min\{ t \, / \, Q(t_n + \hat{t}_{n,1} + t) \, < \, \lambda Q(t_n) \, \} \\ v_n \, > & 0 \end{cases} \tag{15}$$

The definition of $\hat{t}_{n,1}$ is such that $P(t_n + \hat{t}_{n,1}) = Q(t_n)$. The previous paragraph demonstrated the existence of finite times $\hat{t}_{n,2}$. We must verify that this process converges in a finite number of steps, and that the diver can reach the surface. From property 3 and the definition of the sequence, we can infer that $\forall\, n \in \mathbb{N}$, $Q(t_{n+1}) < \lambda\, Q(t_n)$, leading to:

$$Q(t_n) < \lambda^n Q(T)$$

As this sequence tends towards 0 when n tends to infinity, there exists $n_S \in \mathbb{N}$ such that $\forall\, n \geq n_S$, $Q(t_n) < P_S$. Hence, the diver can reach the surface in a finite time, which can be summarized in the following property:

Property 4. *Under assumption 3, the process defined by equation* (15) *to ascend after a dive of duration T converges after a finite number of steps.*

5.3. Speed and experience

We did not explain how the speed v_n was selected, which is the subject of the next section. When the ascent speed is positive, the process converges, but if the speeds are too slow, the compartments will load considerably during the ascent, forcing the diver to remain underwater for a long time. Conversely, if the ascent speed is too rapid, the diver may risk DCS. Regarding this, it is important to understand that **this new proposed process will be calibrated against the experience**, and shall be simulated numerically for many dives before it is considered a usable diving ascent strategy.

6. The speed's importance

6.1. A naive comparison

Before selecting the speeds, the existing decompression models are discussed. Two main classes of decompression models have emerged during the last century of research on the subject, namely:

- The Haldanian class: models that take into account only the dissolved phase of the inert gas. As such, this approach is considered *macroscopic*: these models ignore the bubbles to view the gas from a distance.

- The class considering the free phase, that is the bubbles: on the contrary, these models take a close look to the gas, so their approach is *microscopic*.

As described by B. R. Wienke in his comprehensive book [8], the behaviors dictated by these approaches are directly opposite: the macroscopic approach tends to bring the diver as close as possible to the surface, while the microscopic one tends to keep him or her as deep as possible to crush the bubbles. Wienke unified those two approaches in his Reduced Gradient Bubble Model (RGBM).

As shown by an experiment conducted by the Navy Experimental Diving Unit, and whose results are summarized in [9] and discussed in [10], deep stops can increase the risk of DCS. Obviously, this shows that the control and/or the understanding that we can have of the dissolved phase is better than the control and/or the understanding of the free phase at the moment.

The need to control the ascent rate is common in these two approaches and is presently well established.

This is exactly what the protocol proposed by the COMEX, one of the most advanced and known organisation for commercial diving, does: control the **speed**. Hence, we chose it as a useful starting point.

6.2. The COMEX approach

The COMEX decompression protocol is an ascent curve (see [11]) giving the position of the diver over time, hence imposing a given speed. It is unfortunately given for a rectangular profile: a dive at the constant depth D_0 during the time T. For such a dive, the ascent depth over time is:

$$\boxed{D(t) = \frac{D_0}{1 + \lambda T^{-\frac{5}{2}} D_0^{-3} (t - T)}} \qquad \forall t \in [T; T_f]$$

in which λ is a parameter.

This form does not suggest that this protocol takes into account, like Haldanian ones do, the amount of inert gas in the bodys. This type of saturation protocol depends, in one way or another, on the integral of the pressure over time. To the best of our knowledge (after a few trials to find an integral form), it is not, but perhaps it is simply a matter of finding the right convolution kernel. This decompression protocol, with its associated parameters and range of validity (not reported in this article) has proven very efficient, as it is directly derived from COMEX's knowledge of intensive dives.

6.3. Selection of the speed

Revisiting the correct speed to adopt to reach our decompression ceiling, take the speed suggested by the COMEX protocol... but which one? As was exposed in the previous section this speed varies over the ascent. We know the COMEX protocol only for rectangular profiles, so we do not know the correct speed to adopt if the profile is not trivial. We therefore again consider a rectangular dive of depth D_0 and suggest 2 choices:

Proposal 2 (Decompression ceiling). *After a non trivial dive of duration T, we adopt the COMEX ascent speed for the rectangular dive of duration T leading to the exact same admissible pressure following Bühlmann's approach (with or without our new type of admissible pressure).*

Appendix A.2 demonstrates that such a dive exists. Another choice can be made:

Proposal 3 (Compartments state). *After a non-trivial dive of duration T, we adopt the COMEX ascent speed for the rectangular dive of duration T leading to the closest state of the overall inert gas pressure in the compartments following Bühlmann's approach (with or without our new type of admissible pressure). This closeness is judged in the same N_p norm as the one used in the Bühlmann* *approach selected. Hence, if $\tilde{A}_i$ is the inert gas pressure of compartment i after the non-trivial dive, D_0 is*

$$D_0 = \arg\min_D \left(\sum_{i=1}^{N} |A_i(T) - \tilde{A}_i(T)|^p \right)^{\frac{1}{p}}$$

Determining which of the two proposals is more relevant from a physiological perspective, and especially more robust with respect to DCS, requires more numerical simulations.

We briefly illustrate the adapted Bühlmann protocol for a dive with the following principal characteristics:

- Maximum depth: 40m
- Total duration $T = 40$ minutes
- Depth at T: 20m
- Gradient Factors $l = 0.8$ and $h = 0.8$
- $p = 60$ to be close to the original Bühlmann algorithm
- Targeting the admissible pressure each meter, to simplify the calculation.

Our proposed decompression protocol begins at $T = 40$ minutes: from this time, the profile successively targets the minimum admissible pressure and then waits for it to sufficiently decrease.

7. Conclusion

Understanding DCS and proposing efficient and physiologically based models to generate decompression schedules is a goal far from being achieved, especially integrating personalization parameters. Hence, in the meantime, mathematical artifacts can be used to improve current models, which are widely implemented in diving computers. This article proposed a new form of minimum admissible pressure in the Haldanian framework of parallel compartments. This form, even if not more closely linked to physiology than previous methods:

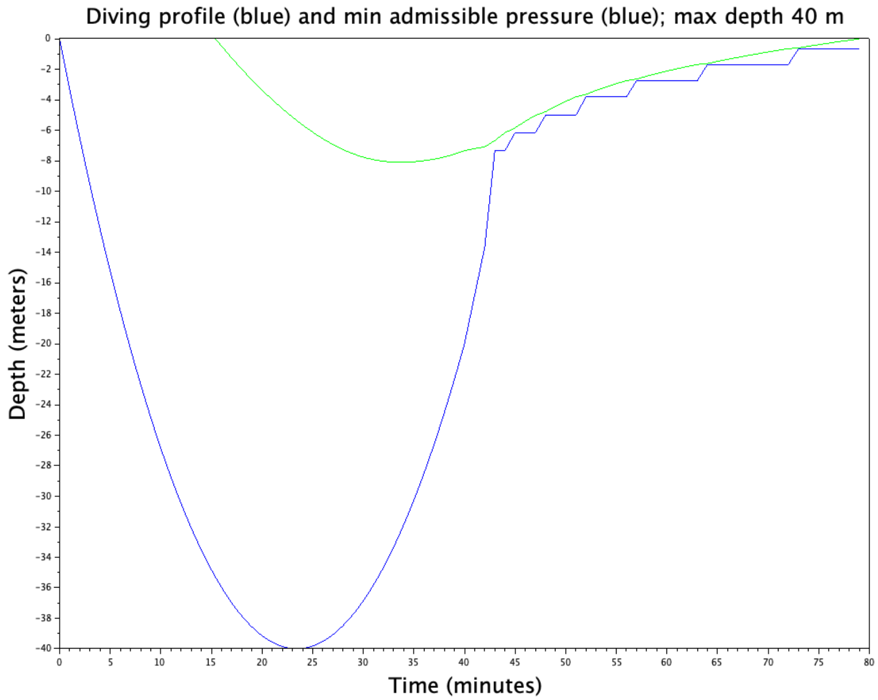

Figure 5: A random dive and our proposed protocol beginning at $T = 40$ minutes

- can be set as close as possible to the original Bühlmann minimum admissible pressure,
- provides more freedom than other methods to adjust decompression parameters,
- enables the modulation of the decompression profile in a different way than Gradient Factors (and can be used along with them),
- is mathematically well defined and has an attractive smoothness property.

We also proposed two other possible methods of introducing personalization leverages. In future research, they could be used as optimization parameters to get as close as possible to empirical protocols, using for example DCS databases linked to physiological data.

As COMEX protocol, based on the control of the ascent speed, has proven its efficiency but cannot be integrated into classical decompression models, we proposed an ascent strategy enabling to unify the selection of the speed with a classical Haldanian approach.

The new form of minimum admissible pressure and the new decompression strategy must be tested and calibrated against real dives. Indeed, as the underlying models are far from representative of any kind of physiology, experience continues to drive them. Nevertheless, we hope that, combined with the increasing knowledge of decompression sickness risk factors, our approach will enable some degrees of personalization, and that new decompression strategies can be tried to aim, as always, for safer dives.

References

[1] A. Bühlmann, Decompression Sickness, no. 10.1007/978-3-662-02409-6-4, Berlin:Springer-Verlag, 1984.

[2] E. C. Baker, Clearing up the confusion about deep stops, Immersed 3 (4) (1998) 23–31.

[3] J. Lautridou, P. Buzzacott, M. Belhomme, E. Dugrenot, P. Lafère, C. Balestra, G. F, Evidence of heritable determinants of decompression sickness in rats, J. Med Sci Sports Exerc 49 (12) (2017) 2433–2438.

[4] R. Eckenhoff, C. Olstad, G. Carrod, Human dose-response relationship for decompression and endogenous bubble formation, Journal of Applied Physiology 69 (3) (1990) 914–918.

[5] D. Cialoni, M. Pieri, C. Balestra, A. Marroni, Flying after diving: in-flight echocardiography after a scuba diving week, Aerosp Med Hum Perform. 85 (10) (2014) 993–998(6).

[6] V. Papadopoulou, P. Germonpré, D. Cosgrove, R. Eckersley, P. Dayton, G. Obeid, A. Boutros, M. Tang, S. Theunissen, C. Balestra, Variability

in circulating gas emboli after a same scuba diving exposure, European Journal of Applied Physiology 118 (6) (2018) 1255–1264.

[7] G. N. Lewis, A mathematical model for decompression, Mathematical Modelling 4 (5) (1983) 489–500.

[8] B. R. Wienke, Basic Decompression Theory and Application, Flagstaff, 2003.

[9] D. J. Doolette, W. A. Gerth, A. Mollerlokken, Redistribution of decompression stop time from shallow to deep stops increases incidence of decompression sickness in air decompression dives, Defence Technical Information Center, Technical report 11-06.

[10] S. Mitchell, Decompression controversies, DAN South Africa (October 2001).

[11] C. Balestra, M. Bedossa, J.-E. Blatteau, R. Brion, B. Brouant, F. Carr, M. Coulange, B. Delemotte, G. Finet, B. Gardette, A. Henckes, Y. Jammes, F. Joulia, V. Lafay, P. Louge, J.-L. Meliet, G. Phan, M. Plutarque, J. Regnard, J.-M. Schleich, F. Schnell, J.-F. Schved, Coeur et Plongée, Ellipses Editions, 2017.

Appendix A. Intermediate properties

This appendix presents intermediate properties that are used in the body of the article. First, the conditions are assessed to ensure that, with the new admissible pressure defined in equation (8), we have $\forall\, t > 0$, $Q(t) < P(t)$.

Appendix A.1. Remaining at a constant pressure

In this paragraph, we drop the index i and suppose that, from $t = 0$ to simplify, the diver remains at the pressure $\hat{P}$. Then the inert gas load in the compartment is, from equation (2):

$$A(t) = A_0 e^{-kt} + R\hat{P}\,(1 - e^{-kt}) \tag{A.1}$$

A tends to $R\hat{P}$ when t tends toward $+\infty$, and as

$$\dot{A}(t) = k(R\hat{P} - A_0)\, e^{-kt} \tag{A.2}$$

A can be increasing or decreasing, depending on the sign of $R\hat{P} - A_0$.

Property 5. *When $P(t) = \hat{P}$ is constant from a given finite time, then the inert gas pressure of each compartment i has the same limit*

$$A_i(t) \underset{t \to +\infty}{\longrightarrow} R\hat{P}$$

by lower or upper values, depending on the sign of $R\hat{P} - A_{0,i}$.

Appendix A.2. Surjectivity of the rectangular profiles application

We try to demonstrate that, given a decompression ceiling $Q(T)$ at the time T, obtained after a random dive (not necessarily rectangular), there exists a rectangular profile of depth D_M leading, after an identical time T, to the same decompression ceiling. For this, we need to start from equation (8) defining Q. From equation (10) we obtain

$$\hat{Q}(t)^p = \sum_{i=1}^{N} \frac{[A_i(t) - a_i l]^p}{[1 + \left(\frac{1}{b_i} - 1\right) l]^p} \tag{A.3}$$

Following equation (12):

$$A_i(t) = A_{i,0} \exp(-k_i t) + P_M R + \frac{P_M R}{k_i t_1} \exp(-k_i t)(1 - \exp(k_i t_1)) \qquad t_1 \le t \le t_2$$

where t_1 is the descent time and P_M is the pressure corresponding to D_M. We take the value of those functions in T and, considering the last equations, we can remark that, T being fixed, the application

$$\mathcal{C} :\ D_M \mapsto Q(T) \tag{A.4}$$

is continuous, and satisfies

$$\mathcal{C}(0) \le P_S \qquad \text{and} \qquad \mathcal{C}(D_M) \underset{D_M \to +\infty}{\longrightarrow} +\infty \tag{A.5}$$

Hence, it is surjective on $[P_S; +\infty[$ and we can deduce the following property:

Property 6. *For a given dive time T, for each descent time t_1 such that $t_1 < T$ and each decompression ceiling $Q(T) \in [P_S; +\infty[$, there exists a finite depth D_M such that $Q(T)$ is the decompression ceiling of the rectangular dive of depth D_M during T with the descent time t_1.*

Please note that D_M (as P_M) is solution of a non-trivial polynomial equation of degree p, that we do not know how to solve explicitely. Nevertheless, this equation has a root, and we can numerically find it quite easily.

Appendix B. Numerical results

The following paragraphs show simulations for various Gradient Factors. The color code for the curves is as follows:

- Thin black line: diving profile. Does not change along the simulations.
- Medium green line: admissible pressure for Bühlmann (BU) with GF 0.3/0.45. Does not change along the simulations.
- Thin red line: admissible pressure for Bühlmann (BU) with the GF considered in the paragraph, if not GF 0.3/0.45. Changes each paragraph.
- Thick blue line: new form of minimum admissible pressure (TD) with the GF considered in the paragraph. Changes each figure.

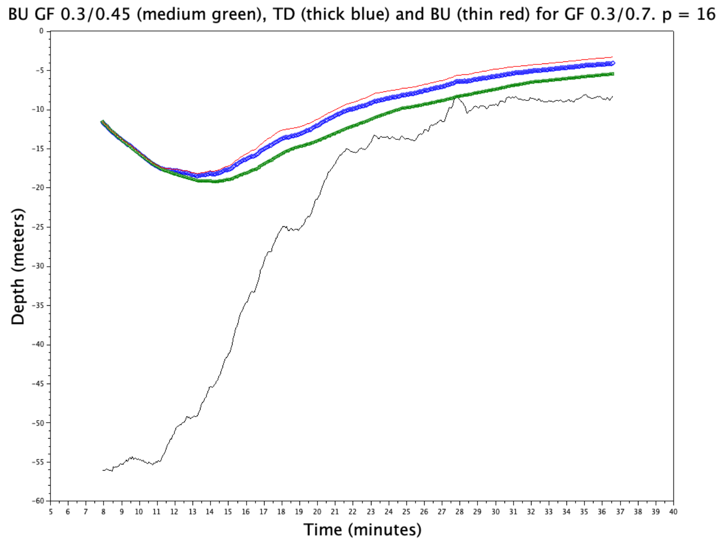

Figure B.6: $p = 16$ for GF 0.3 / 0.7

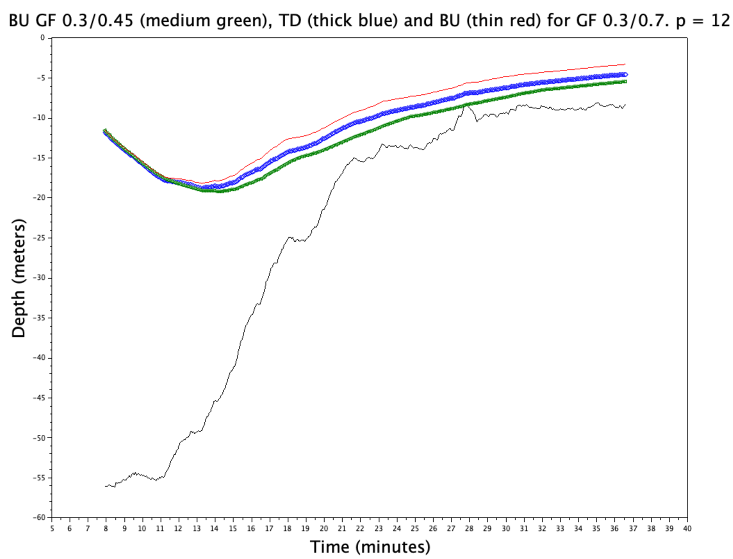

Figure B.7: $p = 12$ for GF 0.3 / 0.7

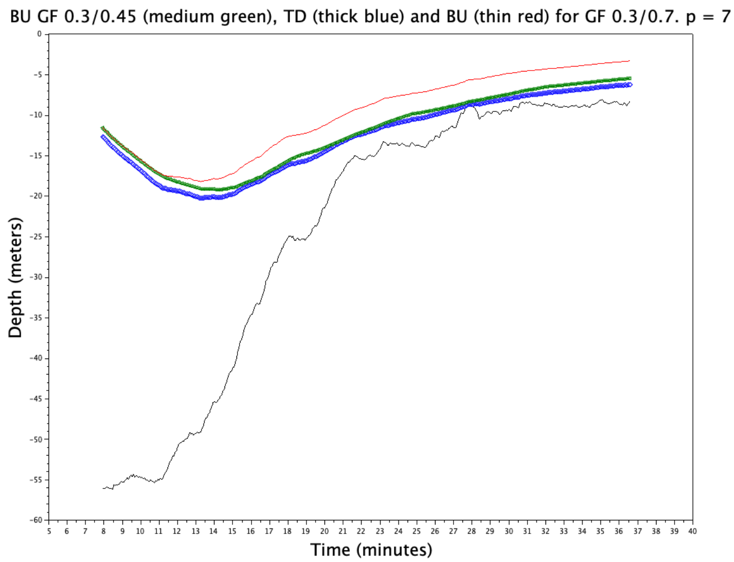

Figure B.8: $p = 7$ for GF 0.3 / 0.7

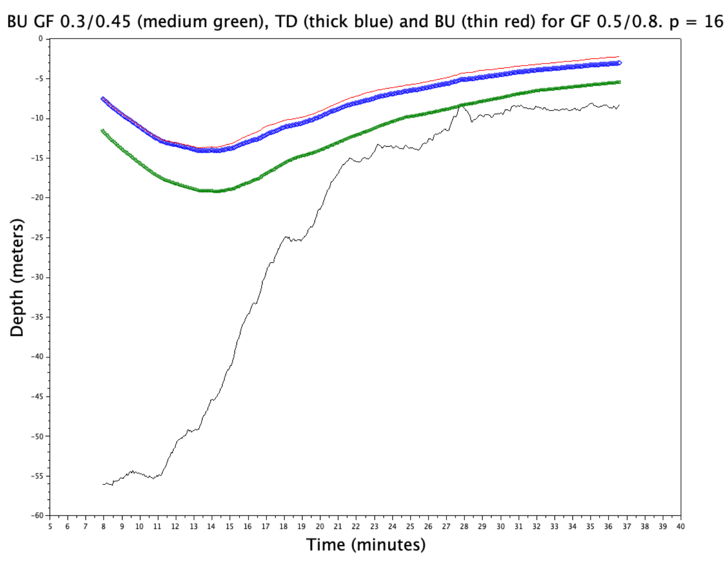

Figure B.9: $p = 16$ for GF 0.5 / 0.8

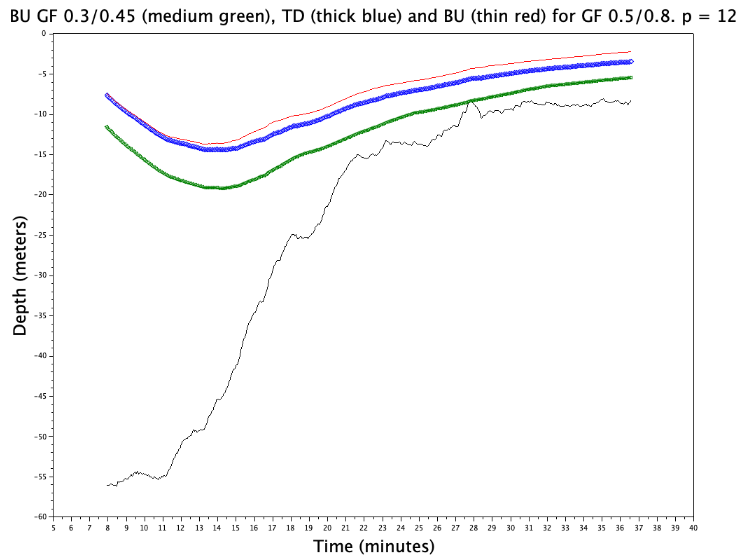

Figure B.10: $p = 12$ for GF 0.5 / 0.8

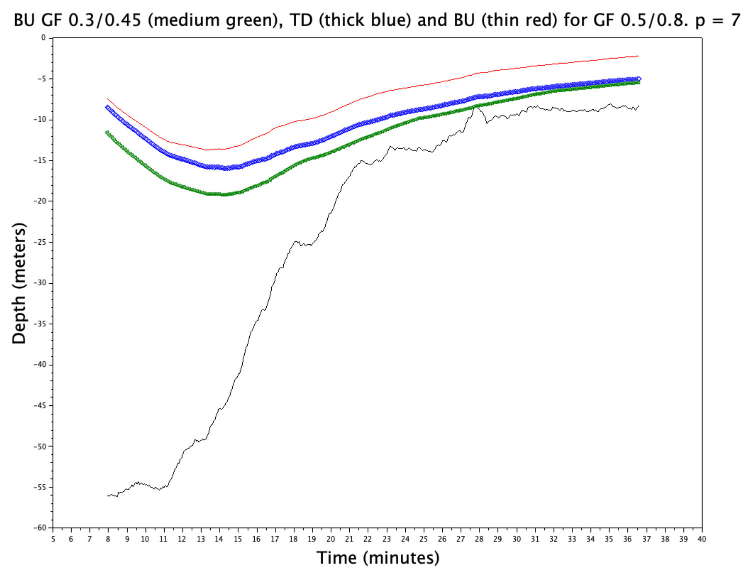

Figure B.11: $p = 7$ for GF 0.5 / 0.8

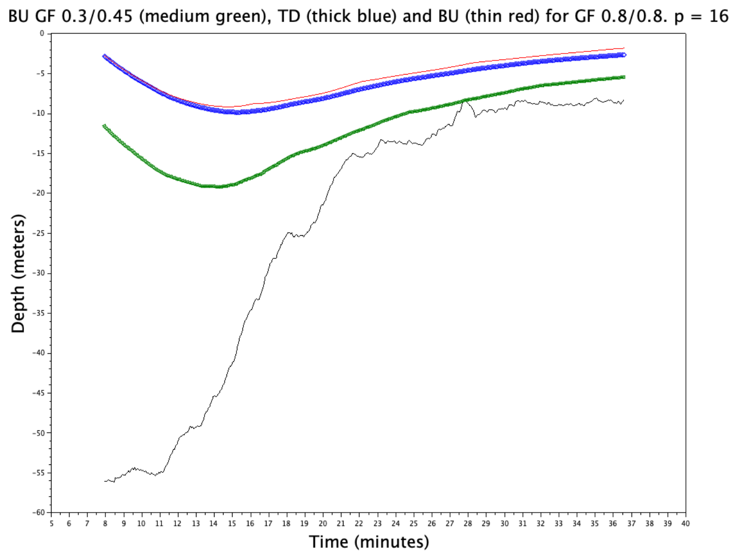

Figure B.12: $p = 16$ for GF 0.8 / 0.8

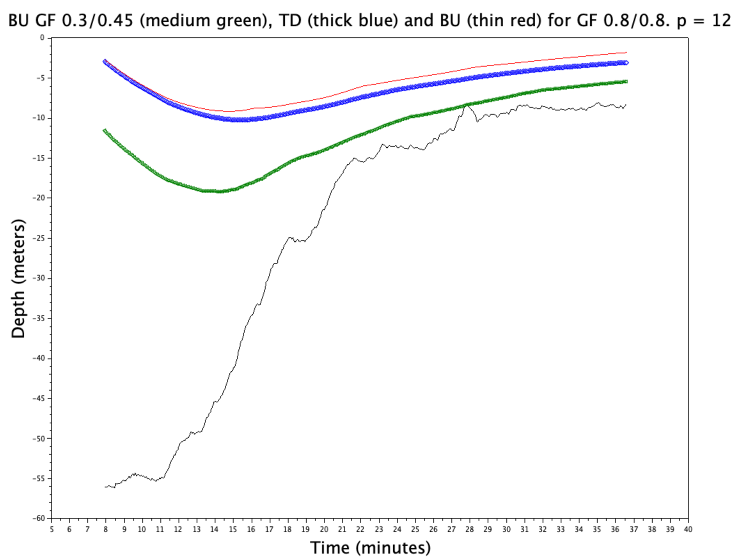

Figure B.13: $p = 12$ for GF 0.8 / 0.8

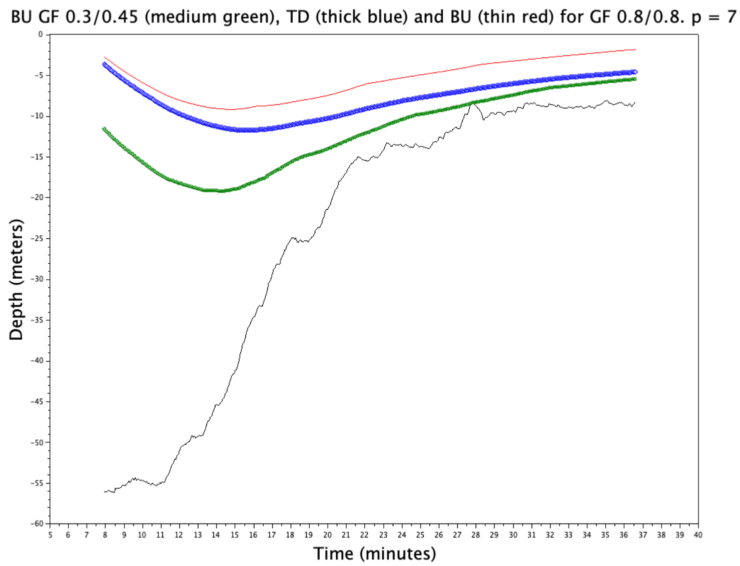

Figure B.14: $p = 7$ for GF 0.8 / 0.8