ON THE INDEPENDENCE OF CONDITIONS IN THE DEFINITION OF LINEAR MAPPPINGS

ASLANBEK NAZIEV[1]

Abstract. We study the (in)dependence of additivity and homogeneity conditions in the definition of a linear mapping between vector spaces over the same scalar field. Unlike other works on this theme, which deal with particular fields like real or complex numbers, we consider the general case. This enables us to obtain an almost complete picture. Namely, for the prime field, the conditions are not independent (additivity implies homogeneity). For the non-prime field of characteristic different from 2, the conditions are independent. For the non-prime field of characteristic 2, the problem remains unsolved.

1. Introduction and preliminaries

By definition, a statement φ is independent of a set of statements $\mathscr{A}$ in a theory $\mathscr{T}$ if neither φ nor $\neg\varphi$ is a consequence of $\mathscr{A}$ in $\mathscr{T}$. In case $\mathscr{A} = \{\psi\}$, they say that statements φ and ψ are independent in theory $\mathscr{T}$. Due to the contraposition law, this means that neither of φ, ψ implies the other in theory $\mathscr{T}$.

In the XX century, the problems of independence attracted sufficient large attention. Great mathematicians K. Gödel and P. Cohen proved very hard theorems about the independence of the axiom of choice and continuum hypothesis from other axioms of set theory ([3], [2], see also [5], [8]. There were also proven results of another type, namely, of consistency (see, for example, [9]). Nevertheless, some problems looking simple remain unsolved. Among such problems is the (in)dependence of conditions in the definition of a linear mapping.

Take as $\mathscr{T}$ the theory $\mathscr{V}(F)$ of vector spaces over one and the same scalar field F. Let U and V be vector spaces over F. As known, a mapping $\varphi\colon U \to V$ is linear iff it satisfies two conditions; (A), additivity: For all u_1, u_2 from U,

2010 *Mathematics Subject Classification.* Primary 54A35; Secondary 03E35.

Key words and phrases. Vector space, additivity, homogeneity, characteristic of a field.

$f(u_1 + u_2) = f(u_1) + f(u_2)$; (H), homogeneity: For every u from U and every λ from F, $\varphi(\lambda u) = \lambda\varphi(u)$.

In the light of previous remarks, the following problem arose: are the conditions (A) and (H) (in)dependent in theory $\mathscr{V}(F)$? Despite the seeming simplicity of the problem, it remains somewhat unsolved. Traditionally, the question "whether (A) implies (H)?" is considered in the context of Cauchy equation theory, predominantly in the cases $F = \mathbb{R}$ or $\mathbb{C}$ under some additional assumptions like continuity (see, for example, [1, 4, 7]). The reverse question ordinarily arises in the context of ODE in exercises connected with the notion of homogeneous equation.

Contrary to these works, we consider here the general case obtaining the following almost complete picture. Namely, in the case of prime F, and only in this case, additivity implies homogeneity, while homogeneity almost surely (excluding, maybe, only one concrete case) does not imply additivity.

Concretely, for the prime field F, (A) implies (H). For the non-prime field F, (A) not implies (H). For the fields of characteristic $\neq 2$, whether prime or non-prime, (H) not implies (A). So, for the prime field F, the conditions (A) and (H) are not independent. For the non-prime field of characteristic, not equal to 2, the conditions are independent. For the non-prime field of characteristic 2, the problem remains unsolved.

2. Main results

2.1. **From (A) to (H).** Recall (see, e. g., [6]) that the prime subfield of a field F is the smallest subfield of F, and that a field F is a prime field if it coincides with its prime subfield. Recall also that every prime field is isomorphic to $\mathbb{Q}$ or $\mathbb{Z}_p(= \mathbb{Z}/p\mathbb{Z})$ for some prime number p.

Theorem 2.1. *Let F be a field, and k its prime subfield.*

If $F = k$, then every additive mapping $U \to V$ with vector F-spaces U, V, is F-homogeneous (and therefore F-linear).

If $F \neq k$, then there exist vector F-spaces U and V and additive mapping $U \to V$, which is not F-homogeneous (and therefore not F-linear).

Proof. Let $F = k$ and $\varphi\colon U \to V$ be additive mapping of the vector F-space U to vector F-space V. Then well known arguments show that φ is F-homogeneous.

[In case $F = k = \mathbb{Z}_p$ for some prime number p, every $\lambda \in F$ has form $\bar{m}$ with $m \in \{1,\ 2, \dots, p\}$, so $\varphi(\lambda x) = \varphi(\underbrace{x + \ldots + x}_{m\ \text{times}}) = \underbrace{\varphi(x) + \ldots + \varphi(x)}_{m\ \text{times}} = \lambda\varphi(x)$.

In case $F = k = \mathbb{Q}$, for every $\lambda \in F$ there exist $m,\ n \in \mathbb{Z}$ such that $n \neq 0$ and $\lambda = \frac{m}{n}$. Due to this we have, first, for every m and n,

$$\varphi(\lambda x) = \varphi\Big(\frac{m}{n}x\Big) = \varphi\Bigg(\Big(\underbrace{\frac{1}{n} + \ldots \frac{1}{n}}_{m\ \text{times}}\Big)x\Bigg) = \underbrace{\varphi\Big(\frac{1}{n}x\Big) + \ldots + \varphi\Big(\frac{1}{n}x\Big)}_{m\ \text{times}} = m\varphi\Big(\frac{1}{n}x\Big).$$

Then, second,

$$\varphi(x) = \varphi\Big(\frac{n}{n}x\Big) = n\varphi\Big(\frac{1}{n}x\Big),$$

whence

$$\varphi\Big(\frac{1}{n}x\Big) = \frac{1}{n}\varphi(x).$$

And lastly, in combination with the first, this yields that

$$\varphi\Big(\frac{m}{n}x\Big) = \frac{m}{n}\varphi(x),$$

that is,

$$\varphi(\lambda x) = \lambda\varphi(x).]$$

Now, consider the case $F \neq k$. Then there exists an $f \in F$ such that $f \notin k$. Clearly, $f \neq 0,\ 1$, and the set $\{1,\ f\}$ is k-independent. Extend this set to a basis B of the k-space F (assuming Axiom of Choice, Zorn's lemma or something like). Define $\varphi\colon B \to F$ by

$$\varphi(b) = f \quad \text{for all } b \in B.$$

The mapping φ has a k-linear extension, say $\tilde{\varphi}$, to the k-space F. This mapping, $\tilde{\varphi}$, is an additive mapping of F-space F to itself which is not F-homogeneous (and therefore not F-linear).

In fact, on the one hand, $\tilde{\varphi}(1) = \tilde{\varphi}(f) = f$ (because $\tilde{\varphi}$ is an extension of φ, and as such it coincides with φ on B). But, on the other hand, if $\tilde{\varphi}$ is F-homogeneous, then there must be $\tilde{\varphi}(f) = \tilde{\varphi}(f \cdot 1) = f \cdot \tilde{\varphi}(1) = f \cdot f = f^2$.

However, $f^2 \neq f$, because

$$f^2 = f \leftrightarrow f^2 - f = 0 \leftrightarrow f \cdot (f - 1) = 0 \leftrightarrow f = 0 \text{ or } f = 1,$$

what is not the case here. □

2.2. **From (H) to (A).** Now we would like to show that homogeneity does not imply additivity. More precisely, for every field F there exist vector F-spaces U and V and a mapping $\varphi\colon U \to V$ such that φ is F-homogeneous but not additive. To do this, look at the characteristic χ of F. First, consider the case with $\chi \neq 2$.

Take $U = F^2$ (two-dimensional arithmetic space over F) and $V = F$. Define $\varphi\colon U \to V$ by the rule

$$\varphi((x,\ y)) = \begin{cases} \frac{xy}{x+y}, & \text{if } x + y \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

So defined φ is F-homogeneous but not additive: In fact,

$$\varphi((1,\ 0)) = \varphi((0,\ 1)) = 0, \text{ but } \varphi((1,\ 0) + (0,\ 1)) = \varphi((1,\ 1)) = \frac{1}{2} \neq 0.$$

It remains to consider the case $\chi = 2$. In this case, we had only success in some particular subcases but not in the whole. Here, we mention only the field $F = \mathbb{Z}_2$. For this F consider the spaces $U = F^2$ and $V = F$ and the mapping $\varphi\colon U \to F$ defined by the rule

$$\varphi((x,\ y)) = \begin{cases} 0, & \text{if } x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, this φ is F-homogeneous. However, it is not additive, for

$$\varphi((0,\ 1) + (1,\ 0)) = \varphi((1,\ 1)) = 1,$$

while

$$\varphi((0,\ 1)) + \varphi((1,\ 0)) = 1 + 1 = 0.$$

In conclusion, we pose a

Problem

Prove or disprove that for every field F of characteristic 2 there exist vector spaces U and V over F and a mapping from U to V which is F-homogeneous but not additive.

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[1] Department of Mathematics, Ryazan State University, Svobody str., 46, Ryazan, 390000, Russian Federation; Ryazan Institute for Education Development, Firsova str., 2a, Ryazan, 390023, Russian Federation; State University for Humanities and Social Studies, Zeleonaya str 30, Kolomna, Moscow reg., 140411, Russian Federation.

Email address: a.naziev@365.rsu.edu.ru